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Do you know what your students are thinking?
An overview of research on knowledge used for teaching

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1. Scene from a College Mathematics Classroom

You are teaching calculus. The class has reached the section on sequences where students are learning the features that indicate whether a particular sequence converges or diverges. You posed the following question and students are discussing it in groups:

Does the sequence $1,\ 0,\ \frac{1}{2},\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{4},\ 0,\ ...$ converge or diverge?

You walk around the classroom, listening to various discussions. When you reach Mia, George, and Lee's group they are in the midst of a lively debate. Mia points to the terms and says, "You see they *both* go to zero so it converges." "No," George chimes in immediately, "because the terms aren't *going* to zero. It jumps around so it can't converge." Before you can respond, Lee slowly brings his thumb and fingers together as he moves his hand to the right and adds, "But the jumps are getting smaller. Can't we just look at the size of the jumps to decide whether it converges or diverges?"

Your mind flies as you try to make sense of what you just heard. Lee's comment stands out – he's actually halfway to Cauchy sequences! George, on the other hand, seems to be confusing convergence and monotonicity, a common issue for students. He might be recalling an example such as 1,0,1,0,… or $(-1)^n$. Mia's comment suggests she's thinking of this sequence as two separate sequences. You respond: "Lee, your question is a great one, and it turns out that in a course called Real Analysis that's the basic idea of a much more complicated concept. For now, let's see if we can work through George and Mia's disagreement. Mia, you seem to be suggesting that there are two sequences here, but George is using the word 'it' as if there's only one sequence. How about I go away for a bit and the three of you try to reconcile these two views? I'll be back to hear more in a few minutes."

This vignette is how one might interact, as a college mathematics instructor, with students who are struggling to understand how to determine the behavior of a sequence. In trying to understand the students' comments and determining how to respond, an instructor would draw upon several different kinds of knowledge. In what follows, we use this vignette to illustrate types of knowledge needed for teaching mathematics. First we provide some history and findings from research about knowledge used in teaching.

2. General overview

While much is known about what instructors do and why, connecting instructors' traits to students' learning is a complex task and no simple relationships are apparent (Darling-Hammond, 1999; Ball, Lubienski & Mewborn, 2001). Traits researchers have examined include instructors' general academic ability, knowledge of mathematics, and specific teaching skills. Common sense suggests that knowledge of mathematics is at the core of effective teaching of mathematics, and it stands to reason that a keen understanding of mathematics is required of instructors. Presumably, a sparse understanding of content is detrimental to effective teaching. It turns out, however, that the relationship between what instructors know and what their students learn is neither clear nor linear.

Although there are many other factors that influence learning (e.g., textbooks, assessments), here we focus on types of teacher knowledge for several reasons. First, knowledge is one of the things that comes immediately to mind when deciding what qualifies someone to teach. Second, instructors' knowledge and the role it plays in shaping teaching practices has been the focus of much investigation (see, e.g., Borko & Putnam, 1996; Schoenfeld, 1999; Sherin, 2002; Shulman, 1986). Finally, some recent research has provided insight into particular kinds of knowledge used in teaching and how these correlate with student achievement. In the following sections, we examine the role knowledge (of various sorts) plays in teaching and instructor effectiveness.

3. Subject Matter Knowledge

a. What is Subject Matter Knowledge?

People who teach typically have taken many courses in their subject and, for some instructional jobs, requirements include completing a program of study (e.g., a major, a master's degree). While not everything is included in a course on a topic, the term *subject matter knowledge* refers to the type of knowledge that it is *possible* to learn in a course, from a textbook, or though other kinds of study. Knowing that the first derivative of a function relates to the slope of that function is an example of subject matter knowledge; so is knowing how to compute the derivative of a particular function such as $y = (x^2 + 7)^3$. But once you study a particular topic, does that mean you are equipped to teach it? For example, if you have taken a calculus course, can you effectively teach calculus to others?

b. How Does Subject Matter Knowledge Matter?

The academic community assumes extensive subject matter knowledge is necessary for teaching and that considerable study beyond the level to be taught is essential to effective teaching (e.g., elementary school teachers have some college-level coursework in mathematics; secondary school teachers usually complete a major in mathematics; and college-level instructors complete graduate-level coursework). While it is natural to assume that what students learn is influenced by what their mathematics instructors know, finding empirical support for this claim has been extremely difficult (Ball, Lubienski, & Mewborn, 2001). Researchers have used the *amount* of mathematics teachers have had opportunities to learn, such as number of courses taken, or number of credits earned, as a measure of subject matter knowledge. It turns out that, using these measures, it is difficult to establish clear relationships between instructors' knowledge and their students' learning of mathematics (Wilson, Floden & Ferrini-Mundy, 2001; 2002).

An often-cited study in this area is Begle (1979). He re-analyzed studies conducted from 1960 through the mid-1970s involving instructor traits and student performance. This meta-analysis found no significant positive relationship between school teachers' highest academic degree, post-bachelor's course credits, or completion of a mathematics major, and student achievement. If only courses beyond calculus are considered, just 10% of the time did teachers' having taken such courses correlate with higher achievement for their students. Even more stunning was the finding that about 8% of the time, having more post-calculus courses was correlated with *lower* levels of student achievement.

More recently, however, Monk (1994) found "positive relationships between the number of undergraduate mathematics courses in a teacher's background and improvement in students' performance in mathematics" (p. 130). The college courses under consideration in Monk's study were those at the sophomore and junior levels. This encouraging picture is tempered a bit by specifics of the findings: taking an additional mathematics course translated into, at most, a 1.2% increase in student performance. It is also important to note that increases of this size were only apparent for the first five undergraduate mathematics courses — completing additional courses beyond five was associated only with a 0.2% gain in student performance.

What these findings suggest is that knowledge of mathematics is not the only or even the best measure predictive of instructor effectiveness and student learning. That is, while knowledge of mathematical content is part of effective teaching, there is more to the story.

4. Pedagogical Knowledge

In addition to knowing the subject, there is much that all teachers, in all subjects, must know in order to provide effective instruction. Such content-independent knowledge is often referred to as *pedagogical knowledge*. This includes, for example, knowledge used to execute basic management routines such as organizing information on a blackboard, knowledge of teaching techniques like the use of "wait time" to allow students sufficient opportunities to respond to questions, and knowledge about the typical behaviors of students at a given grade-level. Like subject matter knowledge, pedagogical knowledge is important. Yet, it also has proved difficult to find strong connections between teachers' pedagogical knowledge and student learning. Much of the work has relied on proxy measures for the pedagogical expertise associated with successful lesson planning, classroom management, and student engagement. Most research has focused on overall student achievement (not mathematics specifically). For example, based on meta-analysis of a collection of studies, researchers found positive correlations between the ranking of school teachers' undergraduate institution and their students' achievement (Wayne & Youngs, 2003). However, such investigations do not tell us what the teachers may have learned at the higher ranked institutions, nor how that might (or might not) contribute to their general pedagogical knowledge about planning, negotiating day to day classroom activity, or motivating students.

Researchers have also sometimes found positive correlations between teachers' scores on licensure exams (Wayne & Youngs, 2003). However, the variety in what is tested and the diversity in the knowledge measured across tests makes it difficult to link specific knowledge to student achievement. Others have found positive correlations between students' achievement and teachers' professional preparation and use of particular teaching routines (Rowan, Correnti, & Miller, 2002).

Another approach used by researchers is to examine correlations between teachers' classroom behaviors and student achievement. For example, Brophy and Good (1986; see also Brophy, 1997) found that, among other things, students learn more in classes where teachers (a) spend time focused on content, (b) create learning activities that are at the appropriate level of difficulty for their students, and (c) get students actively engaged with the material. Morine-Dershimer and Kent (2002) also note that across a collection of studies, teachers' personal beliefs about teaching and learning as well as their practical experience in classrooms correlate with student achievement. An echo of that can be seen in the apparent connections between college instructors' perceptions of online homework and the performance of students on a common final exam (in the technology essay on WebWorK, this volume).

Although these findings provide insight into some features and behaviors of teachers that appear to influence student achievement, we have yet to find ways of identifying and measuring teacher characteristics at a level of detail that links those characteristics to specific elements of pedagogical knowledge. What seems to emerge repeatedly, however, is that the factors are highly inter-connected and, in many cases, highly context- and content-specific. In the context of college mathematics instruction, no extensive research exists on connections among purely pedagogical factors and achievement. However, there are other kinds of knowledge, besides subject matter and pedagogical knowledge, involved in being an effective college mathematics instructor.

5. Mathematics-Specific Teaching-Related Knowledge

Several alternative ways of characterizing knowledge involved in mathematics teaching appear to explain more about student learning than the categories summarized in the previous sections. In this section, we discuss two types of mathematics-specific instructor knowledge: *pedagogical content knowledge* and *specialized content knowledge*.

a. What is Pedagogical Content Knowledge?

Pedagogical content knowledge interweaves knowledge gained from learning the subject with knowledge about teaching that subject (Grossman, Wilson, & Shulman, 1989; Shulman, 1986). This includes, for example, knowledge of typical student difficulties and common ways in which students approach particular mathematical ideas (both unsuccessfully and successfully), along with a curriculum-related knowledge of the scope and sequence of courses of study (e.g., calculus) as well as finer-grained knowledge about examples that are especially illuminating. In mathematics, for instance, instructors know things that are specific to particular topics but were not part of what those instructors were actually taught in their courses (e.g., errors students tend to make when they first learn the quotient rule in calculus or common struggles with the definition of limit). Knowledge of which mathematical topics typically cause students difficulty or how particular examples, explanations, or strategies can be useful in teaching, represent knowledge unique to mathematics instruction yet not obtained exclusively through the study of mathematics itself. Pedagogical content knowledge is in large part what enables instructors to understand why what students write (or say) *makes sense to the student*, even if it is not consistent with standard mathematical usage.

b. Does Pedagogical Content Knowledge Matter?

Since the mid-1980s, when it was named "pedagogical content knowledge," researchers have sought to document the extent to which instructors posses such knowledge and how it relates to classroom practices and student learning. A particularly fruitful line of investigations has come from research associated with "Cognitively Guided Instruction." This has included investigations of school teachers' knowledge of how students think about particular mathematical concepts and examinations of the role such knowledge plays in instructional practices (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). For example, in the vignette we started with here, it was noted that a common student struggle was with confusing convergence and monotonicity. Knowing that students have such confusions and that they are common are parts of mathematics pedagogical content knowledge.

Research findings demonstrate that it is possible to help instructors increase the depth and breadth of their knowledge of student thinking and that such changes in instructor knowledge can lead to positive changes in teaching practices that, in turn, create more opportunities for students to think about and understand mathematical ideas (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996; Fennema & Scott Nelson, 1997). Researchers have also taken this line of work one step farther and examined the consequences of such changes in instructional practice on student achievement. It appears that the more teachers use their knowledge of student thinking while teaching, the more mathematics their students learn (Fennema et al., 1996).

Similar lines of work are just beginning to appear at the college level. While a foundation of such college-level investigations is still being built, researchers have begun to examine the knowledge of student thinking that mathematics graduate students (as teaching assistants and instructors) have for some key concepts from calculus (e.g., limit, derivative). Findings indicate that though graduate students do not necessarily have extensive knowledge of student strategies and difficulties related to these topics when they first begin teaching (Speer, Strickland, & Johnson, 2005), it is possible for graduate students who teach calculus to build rich and detailed understandings of student thinking as they gain teaching experience (Kung, 2010). This is encouraging because it suggests that the large fraction of graduate students who go on to careers that involve college teaching have begun to develop their teaching knowledge before assuming roles as full-time faculty members.

c. What is Specialized Content Knowledge?

In addition to pedagogical content knowledge, there is evidence that there are *special kinds of knowledge of mathematics* that play important roles in how people teach. This type of knowledge is distinct from pedagogical content knowledge because it includes knowledge of mathematics that is not inherently related to understanding student learning. It is also not the content knowledge learned via coursework or texts. This knowledge may come, in part, from the special kind of mathematical work that instructors engage in, a kind of unpacking of mathematical ideas, frequently required for teaching mathematics. Instructors use this knowledge to make sense of students' reasoning, respond to unexpected questions, analyze students' errors, develop meaningful assessments of student learning, and provide mathematically connected and appropriate on-the-spot representations, examples, or explanations for ideas that arise spontaneously in the real-time practice of teaching.

Although there is some debate about how well this conceptualization (which was developed among researchers working with elementary school teachers) applies to teachers at higher grade levels (Speer & King, 2009), it seems clear that such forms of knowledge are certainly (a) mathematical in nature (b) not necessarily part of what can be obtained from advanced mathematical coursework. There are some differences between how mathematics is used in research and how it is used in teaching. Mathematicians value precision, compactness, and elegance in mathematical explanations, while teaching requires that compact mathematical ideas be unpacked (Ball & Bass, 2000; Ferrini-Mundy et al., 2003) into elemental components that are more accessible to students. The issue is not that mathematics experts cannot have this kind of knowledge, but that it is not seen as productive by them – doing mathematics requires the compression or packing of ideas much more often than requiring decompression or unpacking. Mathematics doctoral students may not regularly have the opportunities afforded by teaching experience to develop the particular mathematical insights that are useful for instruction. Moreover, for graduate students who are both learners and instructors, a special set of skills may be needed in order to move flexibly from compressed to unpacked: from a compressed application of a concept, say of functions as elements in a Banach space, as a student during a 9am graduate analysis class to an uncompressed, unpacked, use of the concept of function as an array of connected ideas of variable, co-variation, inputs, outputs, and 2D graphing needed as an instructor to teach a 10am calculus class.

d. Does Specialized Content Knowledge matter?

This kind of knowledge, as well as how it is used in teaching, has received considerable attention in the K-12 mathematics education community (Ball & Bass, 2000; Ferrini-Mundy, Burrill, Floden, & Sandow, 2003; Hill, Rowan, & Ball, 2004, 2005; Hill, Schilling, & Ball, 2004; Ma, 1999). Ma (1999) investigated Chinese and U.S. teachers' knowledge of mathematics in certain domains (subtraction, multiplication, fractions, area and perimeter) and explored what knowledge is linked to the teaching of topics in those domains. For example, teachers who were successful in creating a word problem that modeled a given computation displayed a type of knowledge of fractions and division that is distinct from what is typically gained through regular schooling (at least in the U.S.). That is, the knowledge needed to solve word problems is different from that needed to pose them. Ma concluded that having such problem-posing knowledge enables teachers to make sense of student thinking and contributes to the learning opportunities they can create for their students.

In work on specialized content knowledge, a distinction is made between subject matter knowledge and content knowledge particular to teaching. Such distinctions have been described by Hill, Schilling, and Ball (2004), who have worked to create assessments of teacher knowledge:

> One way to illustrate this distinction is by theorizing about how someone who has not taught children but who is otherwise knowledgeable in mathematics might interpret and respond to these items. This test population would not find the items which tap ordinary subject matter knowledge difficult. By contrast, however, these mathematics experts might be surprised, slowed, or even halted by the mathematics-as-used-in-teaching-items; they would not have had access to or experience with opportunities to see, learn about, unpack and understand mathematics as it is used at the elementary level. (p. 16)

Their items for testing teacher specialized content knowledge include example student solutions where teachers are asked to determine whether the strategy used is one that could be generalized beyond the particular task. For example, teachers might be asked about a non-standard approach to three-digit multiplication and whether such an approach would work with other numbers (from Hill, Rowan, & Ball, 2005):

3. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
x25	x25	x25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

This is not common work for adults who do not teach, however, this work "is entirely mathematical, not pedagogical; to make sound pedagogical decisions teachers must be able to size up and evaluate the mathematics of these alternatives" (p. 388); moreover, "teachers' content knowledge in mathematics, as measured by items designed to be close to the content and its uses that teachers deploy, is positively related to student achievement" (Hill, Rowan, & Ball, 2004, p. 35).

6. Analysis of Knowledge Used in the Vignette

Now, armed with information about different kinds of knowledge, we return to the opening vignette and look at how the instructor drew upon knowledge to make various instructional decisions. Recall that the students were trying to decide whether the sequence converges. There was some disagreement, with one student asserting that it converges, one saying it diverges, and the third wondering whether it was sensible to look at the size of the "jumps" between successive terms to determine the behavior of the sequence.

To productively discuss any mathematics problem with students, you need to know (and understand) the answer to the question yourself. In the vignette, we saw how an instructor could make use of subject matter knowledge – like the knowledge that the sequence converges, the nature of Cauchy sequences, and that both related subsequences, 0, 0, 0, … and $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ converge to zero.

In choosing how to use an example, the instructor has to think about how to structure the way students engage with it. We joined the vignette after that decision had been made, with the students already working in small groups. To make that choice, the instructor relied on familiarity with group work as an instructional strategy: knowing not only that there was an instructional approach called group work, but also some of the techniques needed to use such an approach (e.g., what to say to students to get them to start working in groups, what size group is apt to be most productive). These are all examples of general pedagogical knowledge.

Also, before the vignette, the instructor selected or created the problem. Making an informed choice of problem calls on several different types of knowledge. In this case, one can imagine needing to know something about how students typically think about behavior of sequences and what some common difficulties are that students encounter. Those are all parts of pedagogical content knowledge. Choosing the example also used specialized content knowledge to make sure it had the property that it converged, despite looking similar to the sequence $(-1)^n$ which students might remember as divergent. This particular sequence choice could offer the opportunity to learn the dangers of over-generalizing, in this case, the pitfall of the idea that a sequence that "jumps around" or is not monotone must diverge.

Knowing that the problem would invite debate among students (converges versus diverges), the instructor might have felt that students would have good opportunities to learn the relevant ideas by working with others on the task. That kind of reasoning on the part of the instructor draws on pedagogical content knowledge—the instructor knew that the task was not trivial for the students and that they were apt to have differing opinions about the behavior of the sequence. Absent that knowledge, it might have been quite difficult for the instructor to (correctly) predict that this task was a productive way for students to spend their time.

Having discussed the knowledge the instructor might have drawn upon preceding the vignette, we now turn to the "observed" dialog. When we joined the scene, the students were in the midst of a lively discussion about the problem. The instructor listened (perhaps drawing on some piece of pedagogical knowledge about the merits of student discussion to make the decision not to intervene in what was going on). The first thing that the instructor heard was Mia's comment, "You see they *both* go to zero so it converges." To follow what Mia said, the instructor needed to figure out what Mia meant by "both." Understanding that students typically misinterpret piecewise-defined sequences (and functions) as two separate entities is an example of pedagogical content knowledge. If, on the other hand, the instructor had not designed the problem with this possible student interpretation in mind, making sense of Mia's comment might require the instructor to think about the mathematics in the task in new ways—considering what mathematical features of the task could be described as "both." This type of reasoning might draw on specialized forms of mathematics knowledge, but of a sort that is generated by (and used in) the work of teaching and not in the work of being a mathematician.

The second student, George, responded with, "No, because the terms aren't *going* to zero. It jumps around so it can't converge." While we cannot be sure exactly what George was thinking, it is possible that the sequence, with every other term being zero, reminded him of other sequences he had seen before that "jump around" (and do not converge). To interpret George's statement in this way, the instructor needed to know the other kinds of sequences George would likely have seen that also "jump around" and diverge, including 1,0,1,0,… and $(-1)^n$. Knowing that George had probably seen these is a type of pedagogical content knowledge (knowledge of the curriculum, in particular) and knowing that these sequences do not converge takes knowledge of mathematics. Awareness that George might be over-generalizing from these other sequences also shows a type of pedagogical content knowledge. Many of the most common errors seen in a mathematics classroom involve the over-generalization of correct ideas (an assimilation rather than accommodation – see the essay on Constructivism, this volume, for more on this concept).

The ideas that Mia and George offered were probably sufficient fodder for debate, but the third member of the group, Lee, added another idea to the discussion: "But the jumps are getting smaller. Can we just look at the size of the jumps and decide whether it converges or diverges?" Lee's attention to the size of the "jumps" is quite interesting. He was pointing out a feature of the sequence that connects to some very important advanced mathematics, namely Cauchy sequences. Seeing these connections and recognizing the extent to which Lee's comment does (and does not) capture all the characteristics of a Cauchy sequence makes use of particular knowledge of mathematics content. Although Lee's focus on the diminishing size of the jumps is a valuable observation, that property alone does not ensure convergence of the sequence.

While making sense of Lee's comment took knowledge of mathematics, deciding how to respond drew on other sorts of knowledge. Cauchy sequences and other ideas that Lee's comments bring to mind for a mathematician are beyond the scope of a course in calculus and the students probably were not equipped to work with the ideas that would surface if the instructor pursued Lee's train of thought. Making such a judgment takes knowledge of the curriculum and where students are in that curriculum—all forms of pedagogical content knowledge.

Once the instructor decided that pursuing Lee's idea was not apt to be useful, something had to be done to guide the discussion in a productive direction. Attention needed to be diverted away from Lee's comment without discouraging him from participating in the discussion. The approach taken in the vignette (telling Lee that his observation is great and connected to the content of a later course, but re-focusing the discussion on Mia and George's debate) illustrates a use of pedagogical knowledge. What the instructor did is a technique that can be used in any situation when the conversation is moving in an unproductive direction. Knowing to redirect the conversation relied on content and pedagogical content knowledge, but the way the instructor responded and the attention to Lee's confidence was primarily informed by general pedagogical knowledge about how to orchestrate and sustain discussion.

The decision the instructor made to restate Mia and George's claims and to focus the group on resolving the difference of opinion also took pedagogical knowledge about guiding discussion. The particular phrasing of the instruction to the group also reflected certain kinds of pedagogical knowledge. Instead of saying, "Decide which is right," they are told to try to reconcile the differences. Taking the first approach positions the instructor as the authority, declaring that one of the two answers is correct. In contrast, the approach taken in the vignette conveys that the students' ideas are valued and that there is something worthwhile about engaging in the reasoning of the debate itself.

7. What Does All This Mean for Instructors of College Mathematics?

As noted in the earlier sections, what we know about the roles of knowledge in teaching is based almost exclusively on research in elementary and secondary school contexts. As a result, more information is available to inform the design of professional development for school teachers than is available for post-secondary instruction. That said, the theories that connect certain types of knowledge with teachers' practices and their students' achievement in mathematics are strong and likely apply to teachers of college mathematics as well. In this section we discuss how college mathematics instructors do or could develop important types of knowledge. These approaches are based on the substantial literature base on school teacher professional development as well as information about the context in which mathematics graduate students typically work. A more detailed treatment of these ideas can be found in Kung and Speer (2009).

Instructors develop knowledge in a variety of ways, but two prominent sources are (1) their own experiences as teacher and (2) experiences as a learner about teaching such as professional development programs, reading articles about teaching, etc. Unlike school teachers who typically have substantial access to the latter, college instructors' knowledge development is apt to come mostly from the former, although resources such as these video cases and essays are helping to change that situation.

All teachers engage in three broad categories of work: planning, implementing, and reflecting. By generating plans for instruction, experiencing how those plans unfold in class, and analyzing the extent to which the lesson helped students achieve the learning goals, instructors develop knowledge of the sorts described in preceding sections.

Instructors routinely select problems for students to work on, perhaps as practice with particular skills, or as an assessment of how much learning has occurred, or as an opportunity for students to engage in problem solving. The problems involve particular mathematical ideas and/or skills and instructors can and do learn a tremendous amount of what is difficult for students and how students think by paying attention to students' answers. Those answers might come in written form on a test or be something the instructor hears as students work the problems during class or office hours. No matter the form, each time instructors select and use a problem with students they have created an opportunity to build pedagogical content knowledge around student thinking related to the particular mathematical ideas.

Another common site for developing knowledge useful for teaching are the conversations that instructors have with students. These might occur during class or office hours, but no matter the setting, certain features of such discussions can create especially productive learning opportunities for instructors (while they are creating such opportunities for the students!). Listening to students gives instructors the chance to practice their skills at (1) understanding students' thinking, (2) following the *students'* trains of thought, and (3) doing the specialized mathematical work necessary to determine whether what students are doing and thinking is correct.

The video case activities and associated materials also provide opportunities to enrich your knowledge for teaching. For example, practical applications that focus on developing instructor knowledge of student thinking are described in more detail through the video cases presented here, particularly: *Facilitating Group Work, Leading Whole Class Discussion, Choosing and Ordering Student Work*, and *The Communication Gap*. There are opportunities for reflection and discussion about pedagogical topics in *Grades and Grading* and *Processing Student Feedback* cases.

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